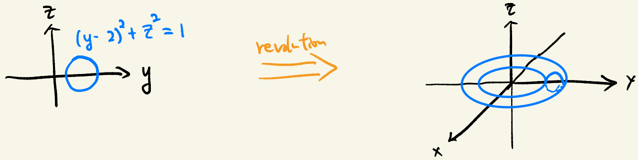
Surface formed by revolution of a curve γ on the x-z plane (in the right half plane)

about the z-axis

e.g.



$$\left(\sqrt{x^2+y^2}-2\right)^2+z^2=1$$

question: 1) what is the equation of the surface so obtained.

Ans: replace y with $r=\sqrt{x^2+y^2}$

2) How to find a parametration of the surface

Ans: let $(y,z)=(\gamma_1(t),\gamma_2(t))$ $t\in[a,b]$ be a parametization of the curve on x-z plane.

A possible parametriation would be

$$(x,y,z)=(\gamma_1(t)\cos\theta,\ \gamma_1(t)\sin\theta,\ \gamma_2(t)) \qquad t\in[a,b] \quad \theta\in[0,2\pi]$$

In our circle $(y-2)^2+z^2=1$, a parametrization would be

$$\begin{cases} y = 2+\cos\varphi \\ z = \sin\varphi \end{cases} \qquad \varphi\in[0,2\pi]$$

The a parametrization for the surface would be

$$\begin{cases} x = (2+\cos\varphi)\cos\theta \\ y = (2+\cos\varphi)\sin\theta \\ z = \sin\varphi \end{cases} \qquad \begin{matrix}\varphi\in[0,2\pi] \\ \theta\in[0,2\pi]\end{matrix}$$

Q1 Show that the area of the surface formed by revolving the graph $z = f(y)$ $0 \le a \le y \le b$ about the z-axis is

$$2\pi \int_a^b t\sqrt{1+f'(t)^2}\, dt$$

A parametrization for the graph is

$$(y, z) = (t, f(t)) \qquad a \le t \le b$$

So a parametriation for the surface is

$$(x, y, z) = \overline{X}(\theta, z) = (t\cos\theta,\ t\sin\theta,\ f(t)) \qquad a \le t \le b \qquad 0 \le \theta \le 2\pi$$

$$\frac{\partial \overline{X}}{\partial t} = (\cos\theta,\ \sin\theta,\ f'(t)) \qquad \frac{\partial \overline{X}}{\partial \theta} = (-t\sin\theta,\ t\cos\theta,\ 0)$$

$$\frac{\partial \overline{X}}{\partial t} \times \frac{\partial \overline{X}}{\partial \theta} = \begin{vmatrix} i & j & k \\ \cos\theta & \sin\theta & f'(t) \\ -t\sin\theta & t\cos\theta & 0 \end{vmatrix} = -tf'(t)\cos\theta\, \vec{i} - tf'(t)\sin\theta\, \vec{j} + t\vec{k}$$

$$\left| \frac{\partial \overline{X}}{\partial t} \times \frac{\partial \overline{X}}{\partial \theta} \right| = \sqrt{t^2 f'(t)^2 + t^2} = t\sqrt{1+f'(t)^2}$$

So

$$\text{Area} = \int_0^{2\pi}\int_a^b t\sqrt{1+f'(t)^2}\, dt\, d\theta$$

$$= 2\pi \int_a^b t\sqrt{1+f'(t)^2}\, dt$$

Rmk: If $z = \sqrt{1-y^2}$ $\quad 0 \le y \le 1$

Area of the hemi-sphere $= 2\pi \int_0^1 t\sqrt{1+\left(\frac{t}{\sqrt{1-t^2}}\right)^2}\, dt$

$$= 2\pi \int_0^1 \frac{t}{\sqrt{1-t^2}}\, dt$$

$$= -2\pi \sqrt{1-t^2}\,\Big|_0^1$$

$$= 2\pi$$

Q2 (Thm of Pappus): let $C(t) = (a(t), b(t))$ $t \in [0,1]$ be a smooth curve in the 1st quadrant of y-z plane.

Let $(\bar{y}, \bar{z})$ be the center of mass of C, L be the length of C

Let S be the surface of revolution of C (about z-axis)

Show that $\text{Area}(S) = 2\pi \bar{y} L$

Ans: $\bar{y} = \frac{1}{L} \int_0^1 a(t) \sqrt{a'(t)^2 + b'(t)^2}\, dt$

A parametrization of S is $\bar{X}(t,\theta) = (a(t)\cos\theta, a(t)\sin\theta, b(t))$

$$\frac{\partial \bar{X}}{\partial t} = (a'(t)\cos\theta, a'(t)\sin\theta, b'(t))$$

$$\frac{\partial \bar{X}}{\partial \theta} = (-a(t)\sin\theta, -a(t)\cos\theta, 0)$$

$$\frac{\partial \bar{X}}{\partial t} \times \frac{\partial \bar{X}}{\partial \theta} = (a(t)b'(t)\cos\theta, -a(t)b'(t)\sin\theta, a(t)a'(t))$$

$$\left|\frac{\partial \bar{X}}{\partial t} \times \frac{\partial \bar{X}}{\partial \theta}\right|^2 = a(t)^2 b'(t)^2 + a(t)^2 a'(t)^2$$

$$\left|\frac{\partial \bar{X}}{\partial t} \times \frac{\partial \bar{X}}{\partial \theta}\right| = a(t)\sqrt{a'(t)^2 + b'(t)^2}$$

Therefore, $\text{Area}(S) = \int_0^{2\pi} \int_0^1 a(t)\sqrt{a'(t)^2 + b'(t)^2}\, dt\, d\theta$

$= 2\pi \int_0^1 a(t)\sqrt{a'(t)^2 + b'(t)^2}\, dt$

$= 2\pi \bar{y} L$

Integration over a parametrized surface S. (With parametrization $\underline{X} = \underline{X}(u,v)$
$(u,v) \in R$)

① Integrating a function f

$$\iint_S f \, dA = \iint_R f \left| \frac{\partial \underline{X}}{\partial u} \times \frac{\partial \underline{X}}{\partial v} \right| du dv$$

② Integration a vector field $\vec{F}$

$$\iint_S \vec{F} \cdot d\vec{A} = \iint_R F \cdot \left(\frac{\partial \underline{X}}{\partial u} \times \frac{\partial \underline{X}}{\partial v} \right) du dv$$

Q3. - S = sphere of radius 1, $P = (0,0,p)$, $|p| \neq 1$

$f : \mathbb{R}^3 \to \mathbb{R}$, $f(X) = \|X - P\|$

Show that $$\iint_S \frac{1}{f}\, dA = \begin{cases} 4\pi & \text{if } |p|<1 \\ \dfrac{4\pi}{|p|} & \text{if } |p|>1 \end{cases}$$

Ans: Use the parametrization:

$$\underline{X}(\theta, z) = \left(\sqrt{1-z^2}\cos\theta,\ \sqrt{1-z^2}\sin\theta,\ z\right) \qquad -1 \le z \le 1 \quad 0 \le \theta \le 2\pi$$

$$f(\theta, z) = \sqrt{1-z^2 + (z-p)^2} = \sqrt{1+p^2 - 2pz}$$

$$\left| \frac{\partial \underline{X}}{\partial \theta} \times \frac{\partial \underline{X}}{\partial z} \right| = 1$$

$$\iint_S \frac{1}{f}\, dA = \int_0^{2\pi}\int_{-1}^{1} \frac{1}{\sqrt{1+p^2-2pz}}\, dz\, d\theta = 2\pi \int_{-1}^{1} \frac{1}{\sqrt{1+p^2-2pz}}\, dz$$

$$= -\frac{2\pi}{p} \sqrt{1+p^2-2pz}\ \Big|_{-1}^{1} \quad (\text{Assume } p \neq 0)$$

$$= -\frac{2\pi}{p}\left[\sqrt{(1-p)^2} - \sqrt{(1+p)^2}\right]$$

If $|p|<1$, then $= -\frac{2\pi}{p}\left((1-p) - (1+p)\right)$

$$= 4\pi$$

If $p>1$, then $= -\frac{2\pi}{p}\left((p-1) - (1+p)\right)$

$$= \frac{4\pi}{p} = \frac{4\pi}{|p|}$$

If $p<1$, then $= -\frac{2\pi}{p}\left((1-p) + (1+p)\right)$

$$= -\frac{4\pi}{p} = \frac{4\pi}{|p|}$$

The remaining case $p=0$, then $f \equiv 1$, so $\iint_S \frac{1}{f}\, dA = \iint_S dA = \text{Area}(S) = 4\pi$

Curl and Divergence (in $\mathbb{R}^3$)

$\vec{F} = (F_1, F_2, F_3)$, $\operatorname{div}\vec{F} = \nabla\cdot\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$

$\operatorname{curl}\vec{F} = \nabla\times\vec{F} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ F_1 & F_2 & F_3 \end{vmatrix} = (\partial_y F_3 - \partial_z F_2 , \partial_z F_1 - \partial_x F_3 , \partial_x F_2 - \partial_y F_1)$

Q4: Let F , G be vector field. f,g be scalar functions,

Show that 1) $\operatorname{div}(\vec{F}\times\vec{G}) = \vec{G}\cdot\operatorname{curl}\vec{F} - \vec{F}\cdot\operatorname{curl}\vec{G}$

2) $\operatorname{div}(\operatorname{grad} f \times \operatorname{grad} g) = 0$

Stoke's thm: - S - surface with boundary curve C

- when you are travelling along C, the surface is on your left.

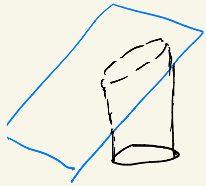
then $\iint_S (\operatorname{curl}\vec{F})\cdot\vec{n}\, dA = \int_C \vec{F}\cdot d\vec{\gamma}$

Q5 $\quad C(t) = (\cos t,\ \sin t,\ \sin t) \qquad 0 \le t \le 2\pi$

Find $\displaystyle\int_C z\,dx + 2x\,dy + y^2\,dz$

1) directly by curve integrals

2) by stoke's thm



Ans: 1) $$\int_C z\,dx + 2x\,dy + y^2\,dz = \int_0^{2\pi} \sin t\, d\cos t + 2\cos t\, d\sin t + \sin^2 t\, d\sin t$$

$$= \int_0^{2\pi} -\sin^2 t + 2\cos^2 t + \sin^2 t\cos t\, dt$$

$$= \int_0^{2\pi} \tfrac{1}{2}(\cos 2t - 1) + (\cos 2t + 1) + \sin^2 t \cos t$$

$$= \quad -\pi \quad + \quad 2\pi \quad = \pi$$

2) $$\text{Curl}\,(z,\ 2x,\ y^2) = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ z & 2x & y^2 \end{vmatrix}$$

$$= 2y\vec{i} + \vec{j} + 2\vec{k}$$

parametrization: $\bar{X}(r,t) = (r\cos t,\ r\sin t,\ r\sin t) \qquad 0 \le r \le 1,\ \ 0 \le t \le 2\pi$

$$\frac{\partial \bar{X}}{\partial r} = (\cos t,\ \sin t,\ \sin t)$$

$$\frac{\partial \bar{X}}{\partial t} = (-r\sin t,\ r\cos t,\ r\cos t)$$

$$\frac{\partial \bar{X}}{\partial r} \times \frac{\partial \bar{X}}{\partial t} = -r\vec{j} + r\vec{k}$$

The integral $$= \int_0^{2\pi}\int_0^1 (2y,\ 1,\ 2)\cdot(0,\ -r,\ r)\ dr\,d\theta$$

$$= 2\pi \int_0^1 r\ dr$$

$$= \pi$$